GETTING STARTED
WITH THE LÉNÁRT SPHERE™

Construction Materials for
Another World of Geometry
Exploring Another World of Geometry on the Lénárt Sphere

Until now you may have experienced any form of geometry other than Euclidean only as a footnote, a topological curiosity, or a historical aside. Maybe a teacher once introduced you to the non-Euclidean properties of spherical geometry, using only pictures from the flat pages of a book or drawings on a flat computer screen.

You probably have not explored spherical geometry with the same depth and rigor that you’ve used to explore plane geometry. You probably have not explored spherical geometry with manipulatives. Maybe you have never explored spherical geometry.

But now you have a Lénárt Sphere.

Why Use the Lénárt Sphere?

Imagine using only spherical surfaces, round books, curved blackboards, and spherical computer screens to study plane geometry!

Using only traditional flat surfaces to learn spherical geometry is just as confusing. Fortunately you can find spherical surfaces everywhere. You can experiment with geometry on balls, oranges, balloons, and tree ornaments. However, you face a problem of accuracy with these handy surfaces. You can measure and draw shapes on a plane with a ruler, a protractor, and a compass, but it is difficult to make accurate constructions and observations on the surface of an orange.
Why Learn Spherical Geometry?

Comparisons to Plane Geometry

Learning spherical geometry gives you new insight into well-known ideas in plane geometry. Each time you pose and solve a problem in one of the two geometries, a similar problem arises in the other. You will see that the same geometric problem may have a different solution in a different context in which there is a different set of rules.

Working on a surface other than the plane gives you the opportunity to confront some of your own mathematical preconceptions and biases. This is exactly what Girolamo Saccheri, Johann Lambert, Carl Friedrich Gauss, János Bolyai, Nikolai Lobachevsky, Bernhard Riemann, and other mathematicians did when they laid the foundations of non-Euclidean geometry.

Applications of Spherical Geometry

Spherical geometry is used by people in many different occupations. Airplane pilots plan routes around a spherical world, chemists discover molecules that can be modeled on a sphere, artists depict spherical figures, and physicists study an entire universe that some believe is curved.

Spherical geometry is also widely used in other branches of mathematics. Many concepts in higher mathematics connect with the concepts of spherical geometry.

Learning About Axiomatic Systems

The second sentence of the United States’ Declaration of Independence begins: “We hold these truths to be self-evident…” and continues with a list of what the authors describe as basic human rights. Almost every system of thought bases its ideas on some fundamental set of beliefs, or axioms. The system of thought is consistent if its axioms, and the theorems deduced from those axioms, do not contradict one another.

Mathematics itself is a system of thought that has gone through a fundamental change in the last two centuries. It has grown from a science built on a few fixed axiomatic systems into one with a multitude of axiomatic systems. There are no longer strict boundaries between the different branches of mathematics, and there are now widened areas of application within and outside the field of mathematics.

You will find that many familiar notions from plane geometry cannot be directly applied to the sphere. For that reason, the axiomatic systems built on the properties of the sphere can be very different from those built on the properties of the plane. Working on the Lénárt Sphere will help you develop an understanding of what an axiomatic system is and how more than one axiomatic system can exist at the same time.

Understanding the Thinking of Others

The type of thinking used in mathematics is an integral part of human thinking. People who learn to be thoughtful about mathematical problems and to accept different approaches in geometry are more likely to apply these ways of thinking to other areas of life.

Investigation into spherical geometry encourages you to compare your ideas with those of others, to reason and argue in a constructive manner, and to look at those with different points of view as partners in finding the truth.
Learning spherical geometry side-by-side with plane geometry may help individuals develop understanding of those with different cultural, traditional, or social backgrounds. Working with different axiomatic systems helps build skills for a diverse and multicultural world and leads to a clearer understanding of how relative all human axioms are. Our hope is that these studies on a sphere may increase tolerance and communication among all of us who live on this globe.

How to Get Started with the Lénárt Sphere

Everything you need to begin exploring the mathematical world of the sphere comes with this Lénárt Sphere Set. We suggest you continue reading and plunge right in. You have our permission to copy pages from this booklet to use in your classroom if it is inconvenient to work directly from the booklet.

The text you use in the study of plane geometry can probably be successfully used with the Lénárt Sphere. Most investigative activities in geometry textbooks translate directly to the sphere, often with startling results.

If you want a ready-made, in-depth curriculum for spherical geometry, you have Non-Euclidean Adventures on the Lénárt Sphere, a collection of blackline activity masters complete with “adventure cards,” student’s guides to the adventures, teacher’s guides to the adventures, a brief philosophical overview of the material, suggestions for using the Lénárt Sphere in the classroom, and a curriculum correlation chart.

Using the Lénárt Sphere Construction Tools

The tools for working on the Lénárt Sphere correspond to the traditional drawing and measuring tools used to study geometry on the plane.

<table>
<thead>
<tr>
<th>Tools for geometry on the plane</th>
<th>Tools for geometry on the sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>You draw and write on a flat surface or on a flat computer screen.</td>
<td>You draw and write on the Lénárt Sphere or on hemispherical transparencies fitted onto the sphere.</td>
</tr>
<tr>
<td>You use a straightedge to draw straight lines, line segments, and rays.</td>
<td>You use a spherical straightedge to draw great circles and arcs of great circles.</td>
</tr>
<tr>
<td>You use a ruler to measure line segments.</td>
<td>You use a spherical ruler to measure great circles and arcs of great circles.</td>
</tr>
<tr>
<td>You use a protractor to measure angles.</td>
<td>You can use a spherical protractor to measure angles.</td>
</tr>
<tr>
<td>You use a compass to draw circles.</td>
<td>You use a spherical compass and a center locator to draw circles.</td>
</tr>
<tr>
<td>If you use a pencil, you can erase your work with an eraser.</td>
<td>If you use a non-permanent marking pen, you can erase your work with a damp cloth or paper towel.</td>
</tr>
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</table>
Sphere
Your Lénárt Sphere Construction Materials Set contains one transparent plastic sphere. This is the surface you will use to investigate spherical geometry. You can draw with non-permanent markers directly on the sphere or on hemispherical transparencies that fit over the sphere. When the sphere is not being held, always rest it on the torus or the spherical ruler. To guard against breakage, never throw or drop your sphere.

Torus
Your set also contains a torus, or donut-shaped surface. The torus serves as a base for the sphere and as another interesting surface on which to experiment.

Transparencies
Each Lénárt Sphere set contains four hemispherical transparencies that fit over the sphere. These transparencies are the “paper” for your spherical “tabletop.” You can draw on these transparencies with non-permanent markers and erase your drawings with a damp cloth or paper towel. You can cut transparencies into various shapes with scissors, and you can connect two hemispherical transparencies with a special hanger to make a hanging sphere.

Sometimes you may want to sketch over a background drawing you want to preserve for future use. In these cases you can make the background drawing on the sphere itself and do additional work on transparencies. Optionally, you can use permanent markers to trace the background drawing on a transparency, then use non-permanent markers to do additional work.

Hangers
Each Lénárt Sphere set contains one plastic hoop or hanger. You can join two transparencies with a hanger to form a hanging sphere. Tessellations make especially decorative constructions to hang in your classroom.

To suspend a pair of transparencies with the hanger, tie a knot near the end of a piece of string and thread the string through the small hole in the hanger or one of the transparencies. The hanger is beveled on each edge so that a transparency can slide on each side. Join the two transparencies by placing one transparency on the torus and fitting the hanger onto its rim. Then fit the second transparency on top. (If necessary, you can use two small pieces of transparent tape to ensure that the hanging sphere does not come apart.)
Spherical Straightedge/Ruler

Use your spherical straightedge or ruler to draw and measure the sphere’s great circles and arcs of great circles. This construction tool also works as a protractor for measuring spherical angles.

The spherical ruler has three “feet” that allow it to stand on a table and hold the sphere. With the sphere cradled in the ruler it is easy to draw a great circle. The two scaled edges trace arcs of great circles; the other edges do not. To measure distance with the spherical ruler, align the arc you are measuring with one of these scaled edges. The markings on the ruler indicate degrees, which are used to measure distance on the sphere. When the spherical ruler is sitting on its feet and holding the sphere, the great circle along its scaled edge has the same inclination as the earth’s equator relative to the sun.

There are several ways of using the spherical ruler as a protractor to measure (or draw) spherical angles. For instance, place the midpoint of the shorter ruled edge at the vertex of the angle. Then line up one side of the angle with this ruled edge. Count the degrees that are along the great circle (longer ruled edge) and that are between the two sides of the angle. You may need to extend the sides of the angles so that they intersect this great circle.

Spherical Protractor

The cap-shaped spherical protractor has four scales of 90°, 10°, 5°, and 1° divisions respectively. This protractor can be very handy, especially for measuring angles in smaller constructions.

Spherical Compass and Center Locator

Your construction set contains a spherical compass and a center locator for drawing circles on the sphere and the transparencies. The center locator has a hole in its center to fix the position of the compass point. To draw a spherical circle, first mark its center on the sphere. Position the sphere on the torus so that the center mark for your circle is on top of the sphere. Place the center locator on the sphere so that the center mark shows through the hole. Then use the spherical compass as you would use its planar counterpart. Just hold the grooved tip between your thumb and forefinger and slowly twist it around.

The spherical compass does not measure degrees as accurately as the ruler. So for constructions where accuracy is important, use the ruler to verify the compass setting. (Warning: Be sure that ink from permanent markers is dry before coming into contact with the center locator.)
Non-Permanent Marking Pens and Collars for Holding the Pens

Your set contains non-permanent marking pens for drawing on the sphere, the torus, and the transparencies. Use water to erase these surfaces, just as you would on an overhead transparency. You can use a spray bottle and paper towels, or you can keep a damp cloth handy.

Occasionally you may want to use permanent overhead markers to preserve an especially nice or important construction on a transparency. Use rubbing alcohol on a tissue or a paper towel to erase permanent markers. (Only write on the sphere with permanent markers made specifically for use with overhead transparencies. Ink from "dry erase" markers, for example, cannot be removed from the sphere.)

The four collars included in your set are designed to hold markers snugly in the barrel of the compass. Simply push the appropriate collar into the barrel of the compass arc until it snaps into place. Then push the marking pen into the collar as far as it will go. To prevent your marking pens from drying out, always remove the pen from the collar and replace its cap when not in use.

The Living Earth Poster and Globe

The Living Earth on the Lénárt Sphere poster includes polyconic projections of the earth's northern and southern hemispheres. You can cut out the polyconic projections and make a spherical map to display as a globe on your Lénárt Sphere. The directions for making the globe are printed on the poster.

Storage Container

The cubical box in which each Lénárt Sphere set was packed is its permanent storage container. Simply remove the loose piece of protective cardboard that surrounded the sphere during shipping, and the box will easily hold a complete set of materials.

Getting Started: Travels on a Sphere

You are already familiar with basic properties of plane geometry. The questions and activities in this section are meant to acquaint you with some basic properties of spherical geometry. You can ponder these activities on your own, discuss them with others, write down your responses, and report them to classmates, friends, and colleagues.

To complete the activities you'll need a few things not included in your Lénárt Sphere set: water for dripping, a damp cloth or paper towel for cleaning the sphere, a piece of string or a rubber band, and a globe (either a commercial globe or one made using your Lénárt Sphere).

Start your travels on the Lénárt Sphere by solving an old riddle. Investigate the riddle by drawing diagrams on your sphere with non-permanent marking pens.

1. A wandering bear leaves home and walks 100 kilometers south. After a rest, she turns west and walks straight ahead for 100 kilometers. Then she turns again and walks north. To her surprise she finds that she arrives back home again. What color is the bear?

If the bear is walking due south on a sphere, is she traveling in a straight path? One of the simplest shapes on a flat plane is a straight line. What does a line look like on a sphere? The next few activities will help you decide. Use your sphere and markers to explain your conclusions. Remember, you can erase constructions on the sphere with a damp cloth or paper towel.

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2. Suppose one morning you leave your front door and start traveling straight ahead. No matter what gets in your way, you find a way to maintain your direction. You travel for a very long time. Describe the path of your trip. Where will you end up?

A great circle is a circle on a sphere that divides the sphere into two equal parts. If the earth were an exact sphere, the equator would be a great circle.

3. Draw two points on your sphere. Use a taut piece of string or a piece of rubber band to find the shortest path between the points. Trace the path with a marker. What would happen if you extended this path?

4. Examine the spherical ruler that came with your Lénárt Sphere. It has many different edges. Decide which edges you can use to trace arcs of great circles.

5. You can measure the arc of a circle in degrees. Your spherical ruler measures distance along a great circle in degrees. Mark two points on your sphere and measure the distance between them.

6. List the similarities and the differences between great circles on a sphere and straight lines on a plane.

7. a) Squeeze some drops of water onto a tilted flat surface and allow the drops to run down the surface. Describe the path of the water.

b) Squeeze some drops of water onto the top of your Lénárt Sphere and allow the drops to run down the surface. Describe what you observe about the path of the water on the sphere.

8. Pilots need to be familiar with spherical geometry so that they can navigate. Imagine you are piloting a plane that takes off in San Francisco and flies directly to Moscow.
b) Find a commercial globe or make a globe with the *Living Earth* poster included in your Lénárt Sphere set. (Making a globe will take some time.) Find the positions of Moscow and San Francisco on your globe. If you’re working on your *Living Earth* globe, use your spherical straightedge and a marking pen to trace the shortest flight route between these points. If you’re working on a commercial globe, use a taut piece of string or a piece of rubber band to trace the shortest flight route between these points. What are some of the places you would fly over if this were your flight path?

c) Compare the two routes. Which would you choose if you were the pilot?

9. Now pretend you are located at the North Pole and need to fly to the South Pole. Which route would you choose to ensure you took the shortest trip?

10. Think again of the long, straight trip that started at your front door. It would be a lonely trip to take all by yourself, so you have decided to bring along a friend. You and your friend travel side-by-side so that you can converse more easily. Explain why it is impossible for both of you to travel in a direct path for the entire trip.

11. Imagine that a pair of railroad tracks extends all the way around the earth. Decide whether the railroad tracks could represent parallel lines.

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**Summarizing Your Travels: Some Basic Properties Revisited**

This is a good place to stop and summarize some of the discoveries you have made so far. Use a chart like the one shown below to record your thoughts and to compare geometry on the plane with geometry on the sphere.

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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>The parts into which two points divide a line</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The number of lines that pass through any two different points</td>
<td></td>
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Getting Started: A Construction on the Sphere

You learned that some basic properties of plane geometry do not hold true on a sphere. What about the more complex geometric ideas that depend on these properties? As you can imagine, once the basic properties change, the rest of the geometry is fundamentally changed.

The two constructions in this section will give you a chance to explore some essential differences between planar and spherical geometry. Along the way, you will become familiar with all the spherical construction tools. For the constructions on the plane, you’ll need paper and the familiar planar construction tools—a ruler, protractor, and compass—or a computer construction tool. For work on the Lénárt Sphere you’ll also need a damp cloth or paper towel for cleaning the sphere.

Construction on the Plane

A sheet of paper or a computer screen can represent an infinite flat plane. Use one of these and some planar construction tools to follow the five construction steps given.

Step 1
Draw a point O. Then use a protractor to draw three rays that begin at point O and that form three angles, each measuring 120°.

Step 2
Use a compass to draw a circle with point O as its center. Mark the points where the circle intersects the three rays. Connect all three of these points of intersection using a straightedge to form an equilateral triangle.

Step 3
Draw three more circles with center O, using different radii. Use any size radii you like. For each circle, find the points of intersection with the three rays and connect them to form an equilateral triangle.

Step 4
Measure the sides of all your equilateral triangles with a ruler. Label each side with its length.

Step 5
Measure the angles in all of your equilateral triangles with a protractor. Label each angle with its measure.

Construction on the Sphere

Follow the given steps to make a matching spherical construction on your Lénárt Sphere.

Step 1
Draw a point O. Then use your spherical protractor to draw three arcs of great circles that begin at point O and that create three angles, each measuring 120°.

Step 2
Use your spherical compass to draw a circle with point O as its center. Mark the points where the circle intersects the three arcs. Connect all three of these points of intersection using your spherical straightedge to form an equilateral spherical triangle.

Step 3
Draw three more circles with center O, using radii that measure 30°, 60°, and 90°. For each circle, find the points of intersection with the three arcs and connect them using your spherical straightedge to form an equilateral spherical triangle.

Step 4
Measure the sides of all your equilateral triangles with your spherical ruler. Label each side with its length.

Step 5
Measure the angles in all of your equilateral triangles with your spherical protractor. Label each angle with its measure.
Reflecting on Your Constructions: Some Basic Ideas Revisited

Use the constructions you just created to help answer the given questions. Be aware that some of these questions may cause many days of pondering!

1. What is the sum of the measures of the angles of a triangle?
2. Are equilateral triangles always similar? (Remember, polygons are similar if their corresponding sides are proportional and their corresponding angles are congruent.)
3. What is the relationship between the angles of a triangle and its size?
4. Is there a largest circle?
5. Is there a largest triangle?
6. Can two lines be parallel?
7. If you halve the diameter of a circle, how do you change its circumference?
8. Is it possible for a circle to become so large that it is a line?

Answers to Getting Started Activities

Getting Started: Travels on a Sphere

1. The bear is white because she lives at the North Pole and must be a polar bear. Her path on the sphere describes a three-sided spherical shape. All of the three sides are arcs of great circles, because “walking south” and “walking north” can only mean walking along longitudes of the earth along great circles. “She turns west and walks straight ahead” means that she walks along another spherical straight line, that is, a great circle. If the wording were “she walks continuously west,” then she would walk along a latitude of the earth along a smaller circle.

2. Your path will follow a circle whose center is also the center of the earth. Such a circle is called a great circle. You will eventually end up at home again.

3. The path is an arc of a great circle. If you extend it, you will draw the entire great circle. Paths along great circles and straight lines travel the shortest distance between two points. These are the most direct paths possible on their respective surfaces.

4. The edges of the spherical ruler that trace arcs of great circles have ruled markings. The unmarked edges trace arcs of smaller circles.

5. Answers vary depending on the location of the points. The distance between any two points on the sphere cannot be more than 180°.
6. Great circles have finite length, whereas straight lines are infinite in length. A great circle divides the sphere into two finite, congruent sections, whereas a straight line divides the plane into two infinite sections. Both great circles and straight lines connect points with the shortest possible path, and both shapes trace the most direct path in any direction.

7. a) If all goes well, the water will trace the path of a straight line. 
   b) The water should trace an arc of a great circle.

8. a) The plane would fly over the western and central United States, eastern Canada, the Atlantic, United Kingdom, northern Germany or Denmark, the Baltic Sea, and Russia. 
   b) The shortest route is the polar route. The plane would fly over the northwestern United States, western Canada, the Canadian Arctic, Greenland, Scandinavia, and Russia. 
   c) The shortest route is that found by using a great circle on a sphere.

9. There are infinitely many great circles that pass through any pair of pole points, so there are infinitely many routes to pick that are equally short. You might choose any one of these.

10. If both you and your friend are traveling in direct paths, your paths must follow great circles. Great circles on a sphere always intersect. So either you and your friend eventually run into each other (or start stepping on each other’s feet) or one (or both) of you follow a path of a circle that is smaller than a great circle.

11. Presumably the two tracks never intersect. Since every pair of great circles intersects, it follows that both tracks cannot lie on great circles. Therefore the tracks cannot represent parallel lines (great circles) on the sphere.

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**Summarizing Your Travels: Some Basic Properties Revisited**

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<tr>
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<td>The shortest path that connects two points is a straight line segment.</td>
<td>The shortest path that connects two is an arc of a great circle, or a spherical segment.</td>
</tr>
<tr>
<td>Extending a line forever</td>
<td>You can do this on the plane. Because the plane extends indefinitely, you can extend a straight line forever.</td>
<td>You cannot do this on the sphere. A line on a sphere is a great circle which meets itself again and has a finite length.</td>
</tr>
<tr>
<td>The number of lines that pass through any two different points</td>
<td>Exactly one straight line passes through any pair of points on the plane.</td>
<td>Exactly one great circle passes through any pair of points on the sphere unless they are opposite points, or pole points. In this case there are infinitely many great circles that pass through the two points.</td>
</tr>
<tr>
<td>The number of lines parallel to a given line that pass through a point not on the given line</td>
<td>Euclid’s parallel postulate states that there is exactly one such line on the plane.</td>
<td>The parallel postulate does not hold on the sphere. Parallel great circles do not exist on the sphere. A version of the postulate for the sphere might state: Given a great circle and a point not on the great circle, there is no great circle through the point that is parallel to the given great circle.</td>
</tr>
</tbody>
</table>
Reflecting on Your Constructions: Some Basic Ideas Revisited

These questions can lead to discoveries about a great many differences between planar and spherical geometry. The answers given here are brief. There is much more to say. You can find more detailed answers and definitions in Non-Euclidean Adventures on the Lénárt Sphere.

1. The sum of the measures of the angles of a triangle on the plane is always 180°. The sum of the measures of the angles of a triangle on the sphere varies according to how much of the sphere the triangle covers. The smallest sum is for a very small triangle that is practically “flat.” The measures of the angles of this triangle sum to slightly more than 180°.

If you decide the triangle with vertices on the same great circle is the “largest” triangle, then the largest angle measure sum is 540°. If you don’t allow the vertices of a triangle to lie on a great circle, then the largest angle measure sum approaches but never equals 540°. It is possible to argue that a triangle can have an angle measure sum of as much as 900°! It depends on how you define a triangle on the sphere.

2. Similar shapes that are not congruent do not exist on the sphere. Although polygons on the sphere can have corresponding sides in proportion, their corresponding angles will not be congruent. A shape on a sphere distorts as you change its scale. Although all equilateral triangles on the plane are similar, equilateral triangles on the sphere are never similar unless they are congruent.
3. The angles of triangles on the plane are independent of the size of the triangle. This is not true on the sphere (see the answer to question 1).

4. On the plane it is always possible to draw a circle that contains (is larger than) a given circle, so there is no largest circle. On the sphere there is no circle that can contain a great circle, so a great circle is the largest circle.

5. On the plane it is always possible to draw a triangle that contains (is larger than) a given triangle, so there is no largest triangle. On the sphere no triangle can contain the large “triangle” that has all three vertices on a great circle, so this circular triangle is the largest triangle—if you allow the vertices of a triangle to lie on a great circle.

6. On the plane you can draw infinitely many pairs of parallel straight lines. On the sphere there are no parallel great circles.

7. If you halve the diameter of the circle on the plane, the circumference also halves. If you halve the diameter of a circle on the sphere, the circumference of the smaller circle will be greater than half of the circumference of the larger circle. The circumference of a circle on the plane is always in the same ratio to its diameter. This ratio, \( \pi \), is constant, no matter what size the circle is. On a sphere the ratio of the circumference to the diameter is not constant but varies according to the size of the circle. For a small circle the ratio is slightly less than \( \pi \); for a great circle the ratio is exactly \( 360°/180° \), or 2. Thus the ratio of the circumference to the diameter in any spherical circle is a number less than \( \pi \) and greater than or equal to 2.

8. If you travel on the plane along a very, very large circle, you may feel like you are traveling in a straight line but your path is always slightly curved. On a sphere, however, a great circle is the spherical equivalent of a straight line. Thus if you travel along a great circle, you are traveling in a line.