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ADVENTURE 2.1

How many points can two lines share?

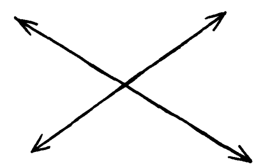
When two distinct lines intersect on the plane or on the sphere, they meet at one or more points.

- Investigate the points of intersection of two straight lines on the plane.
- Investigate the points of intersection of two great circles on the sphere.
- Explain your observations about parallel lines on the plane and on the sphere.

Construction on the Plane

Step 1 Draw a straight line. Label it *l*.

Step 2 Try to draw another straight line that has no point in common with line *l*. Label it *a*.



Step 3 Try to draw a straight line that has exactly one point in common with line *l*. Label it *b*.

Step 4 Try to draw a straight line that has exactly two points in common with line *l*. Label it *c*.

Step 5 Try to draw a straight line that has more than two points in common with line *l*. Label it *d*.

Investigate

1. Which constructions were possible on the plane?

2. Which of your lines are parallel? Why?

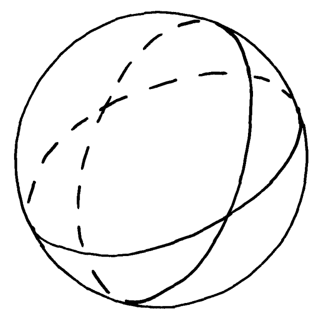
3. Describe all the different ways in which two distinct lines can intersect on the plane.

Make a Guess

4. Will your conclusions be the same for great circles on a sphere?

Construction on the Sphere

5. Perform the same steps on the sphere that you performed on the plane, replacing the straight lines with great circles. Keep track of which constructions are possible on the sphere.



Investigate

6. Describe all the ways in which two distinct great circles can intersect on the sphere.

7. Can two great circles ever be parallel?

Compare the Plane and the Sphere

8. See how many observations you can make about the intersection of two straight lines on the plane and the intersection of two great circles on the sphere. Record them on a comparison chart like the one at right. Add as many rows as you need.

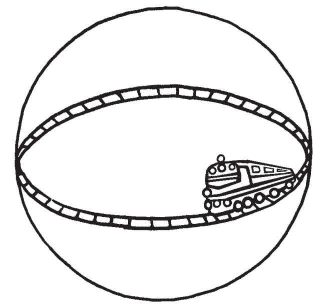
Intersection of two lines

On the plane	On the sphere

9. Do you think the intersection of two lines is simpler on the plane or on the sphere? Which case is more intriguing? Why?

10. Now try to reverse your argument. Give reasons why the intersection of two lines is simpler or more intriguing on the surface you *didn't* choose above.

Explore More



11. Imagine that it is possible for a pair of railroad tracks to extend all the way around the earth. Can the railroad tracks represent parallel lines?

12. Parallel lines on the plane are always the same distance apart. Draw a great circle on your sphere. Then draw a different figure that is always the same distance from your great circle.

a. Describe this figure.

b. Decide if the figure could be a great circle.

13. A boat travels in a such a way that it is always 50 km from the equator. Explain why the boat is not traveling in the most direct path between two points.

14. Euclid was a mathematician from ancient Greece who is famous for being one of the first to organize the ideas of geometry. In his treatise titled *Elements,* Euclid lists a set of **axioms** for geometry. Euclid's axioms were statements that he believed were so obviously true that he was willing to accept them without proof. Almost two and a half thousand years later, we still base plane geometry on Euclid's axioms. However his last axiom, commonly called the parallel postulate, has always been open to debate. Here is one form of Euclid's parallel postulate: Given a straight line and a point not on this straight line, you can draw only one straight line through the given point that is parallel to the given straight line.

a. On a piece of paper draw a straight line and a point not on the line. Draw as many lines as you can through the point that are parallel to the first line. Use your drawing to explain why Euclid's parallel postulate makes sense on the plane.

b. Rewrite the parallel postulate for the sphere by replacing the words *straight line* with *great circle.* Then make a construction on your sphere similar to the construction you just made on the plane. Now explain why Euclid's parallel postulate does not make sense on the sphere.

c. Write your own parallel postulate that is true for geometry on a sphere.

15. Describe all the ways that three distinct great circles can intersect.

How many points can two lines share?

Student Audience: Middle School/High School

Prerequisites: Students should know the terms line and great circle.

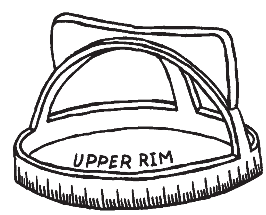
Class Time: 25–40 minutes.

Construction/Investigation on the Plane

In Step 2, students can use the parallel edges of the straightedge. Step 4, of course, cannot be constructed. The only construction in Step 5 is found in two coincidental lines with infinitely many points in common.

Two distinct lines are either parallel to each other, with no common point, or intersecting, with exactly one point of intersection.

Construction/Investigation on the Sphere



Students may be surprised to learn that parallel great circles (with no common point) do not exist. Sometimes they construct a parallel circle that is not a great circle, using the upper unscaled rim of the base hoop of the spherical ruler. To help students see this, ask them to stretch a taut string between two points of a "false" circle. On the sphere, Step 4 is the only possible construction. The only solution in Step 5 is found in coincidental great circles, just as it is with lines.

Two distinct great circles always intersect in exactly two points.

Compare the Plane and the Sphere

Intersection of two lines

On the plane	On the sphere
Two distinct straight lines with no point of intersection are called parallel lines.	Two distinct great circles can never be parallel to each other.
Two distinct straight lines with exactly one point in common are called intersecting lines.	Two distinct great circles can never have exactly one point of intersection.
Two distinct straight lines can never have more than one point of intersection.	Two distinct great circles always have exactly two points of intersection.

9. We must consider three cases on the plane: parallel, intersecting, and coincident lines. On the sphere, however, we must only consider two because parallel great circles do not exist. Therefore in this investigation, the sphere is simpler than the plane.

Explore More

11. Presumably the two tracks never intersect. Because every pair of great circles intersects, it follows that both tracks cannot lie on great circles. Therefore both of the tracks cannot represent great circles on the sphere.

12. The figure is a circle with a radius between 0° and 90°. If the distance in question measures 0°, then the circle is the original great circle with radius 90°. If the distance measures 90°, then the circle is the pole point of the original great circle.

13. The boat isn't traveling in the most direct path between two points because a line that is always at a 50-km distance from the equator must be a circle smaller than a great circle.

14. **c.** Answers should sound something like this: Given a great circle and a point not on the great circle, there is no great circle through the point that is parallel to the given great circle.

As you see, there is one straight line through a given point that is parallel to a given straight line on the plane. There is no great circle through a given point that is parallel to a given great circle on the sphere. Logically, the third possibility is a surface with some kind of simplest line on it, which adheres to the following property: There is more than one simplest line through a given point that is parallel to a given simplest line. Can we find such a surface? This question was first answered in the nineteenth century by Carl Friedrich Gauss, János Bolyai, Nikolai Lobachevsky, and others. Indeed, we can find such surfaces, but these are a bit more complicated and less demonstrative than the plane with the straight line or the sphere with the great circle. For example, if we replace the straight lines of the plane or the great circles of the sphere with certain arcs of certain circles as the simplest lines, then we can build up geometries of the third kind.

15. If the three great circles are concurrent, then they have two points of intersection. If they are not concurrent, then there are six points of intersection on the whole sphere.